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On the hydromagnetic stability of a rotating fluid annulus

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A non-dissipative fluid rotates uniformly in the annular region between two infinitely long cylinders and is permeated by a magnetic field varying with distance from the axis of rotation. The hydromagnetic stability of this system is examined theoretically. When the magnetic field is azimuthal the system can always be rendered stable to axisymmetric disturbances by sufficiently rapid rotation (Michael 1954). Unless the magnetic field everywhere decreases with radius, however, the system may be unstable to *non-axisymmetric* disturbances even when the rotation speed exceeds a typical Alfvén speed by many orders of magnitude. 'Slow' hydromagnetic waves, akin to those invoked in a recent theory of the geomagnetic secular variation (Hide 1966), may then be generated by the spatial variations of the magnetic field. All unstable waves so generated propagate against the basic rotation, i.e. 'westward', when the field is azimuthal, and this property is in fact remarkably insensitive to variations in both magnitude *and* direction of the imposed field.

1. Introduction

Following a recent suggestion (Hide 1966) that the slow westward drift with time of the non-dipole geomagnetic field and the general time scale of the geomagnetic secular variation may be a manifestation (in part, at least) of free hydromagnetic oscillations of the earth's liquid core there has been considerable interest in the propagation of hydromagnetic waves in a bounded rotating fluid. Oscillations of an inviscid, perfectly conducting, incompressible and homogeneous fluid rotating between two concentric spheres in the presence of a predominantly azimuthal magnetic field are of particular interest in this connexion. Analysis of the 'thin-shell' case, in which the radii of the spheres are nearly equal (Hide 1966; Stewartson 1967), reveals the existence at 'rapid' rotation rates (i.e. low values of the parameter

$$Q \equiv V/\Omega\lambda, \tag{1.1}$$

where V is a typical Alfvén speed,‡ Ω the angular velocity of rotation and λ a

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‡ This is the speed at which hydromagnetic waves would propagate if the fluid were not rotating. $V = B/(\mu\rho)^{\frac{1}{2}}$, where B is the magnetic field, ρ the fluid density and μ the magnetic permeability (see, for example, Shercliff 1965). Rationalized mks units will be used throughout.

typical wavelength) of two distinct classes of wave motion. In addition to the familiar planetary waves, first examined by Rossby (1939), there are 'hydromagnetic-planetary' waves which propagate *eastward* very slowly compared with the Alfvén speed with typical phase velocities of order QV. Using the results of his thin-shell analysis Hide (1966) argued that in a *thick* spherical shell such as the earth's liquid core (for which $Q \sim 10^{-3}$) those slow hydromagnetic waves characterized by quasi-two-dimensional motions (in which fluid filaments parallel to the rotation axis move as coherent units) would propagate *westward*. While subsequent analyses (Stewartson 1967; Malkus 1967*a*; Rickard 1970; Acheson 1971) do not appear to deny that the boundaries of a thick spherical shell may constrain the slow '*filamentary*' wave motions to propagate westward in this way, they make clear that other, quite different, *three-dimensional* modes of hydromagnetic-planetary wave propagation are possible. Malkus (1967*a*), for example, examined various modes of oscillation in a full sphere, taking an azimuthal magnetic field profile corresponding to a uniform electric current along the rotation axis. He found waves propagating both east and west with, on balance, no significant preference for either direction. He suggested, however, a possible mechanism for the selective excitation of westward-propagating waves. Laboratory experiments on (non-hydromagnetic) flows induced in rotating spheroids by forced precession indicate the presence of azimuthal flows with sharp discontinuities in slope. Quasi-two-dimensional wave-like instabilities form on such discontinuities and a hydro*magnetic* theory (Malkus 1967*b*) shows that at low values of the parameter Q hydromagnetic-planetary waves generated in this way propagate westward relative to the velocity at the slope discontinuity.

In this paper we continue the search for plausible westward selection mechanisms by examining the generation of 'slow' hydromagnetic waves in a rotating fluid by instabilities resulting from spatial variations in the magnetic field. Mathematical difficulties are greatly reduced by replacing the spherical boundaries by concentric *cylinders*, and the concomitant simplifications permit investigation of a very wide range of magnetic field profiles. While the character of the *quasi-two-dimensional* oscillations is especially sensitive to the geometry of the boundaries (Hide 1966) it is not *a priori* evident that we may not learn of the qualitative properties of the more complex oscillations of a spherical system in this way. Nevertheless (and in spite of the encouraging result, which we shall prove in §3, that all unstable modes generated by the spatial variations of an azimuthal magnetic field propagate westward relative to the rotating fluid), the analysis will be, for the most part, presented as a straightforward extension of the *axisymmetric* stability analyses of Michael (1954), Velikhov (1959) and Chandrasekhar (1961). Discussion of the extent to which this selection mechanism and the westward drift of the earth's magnetic field may be related necessarily involves concepts that are under constant revision as new evidence regarding the earth's interior comes to light, and is best postponed for inclusion in a future paper.

2. Mathematical formulation

To investigate the hydromagnetic stability of an inviscid, perfectly conducting, incompressible and homogeneous fluid rotating with angular velocity $\mathbf{\Omega}$ it is convenient to choose a set of uniformly rotating cylindrical polar co-ordinates (r, θ, z) relative to which the fluid is at rest. The imposed magnetic field

$$\mathbf{B}_0 = \{0, B_\theta(r), B_z(r)\}$$

varies in both magnitude and direction with distance from the rotation axis and the fluid is bounded by two infinitely long cylinders $r = r_1$ and $r = r_2$.

The appropriate MHD equations relative to the rotating co-ordinate system are

$$\partial \mathbf{u}/\partial t + (\mathbf{u}.\nabla)\mathbf{u} + 2\mathbf{\Omega} \wedge \mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\mu\rho}(\nabla \wedge \mathbf{B}) \wedge \mathbf{B}, \tag{2.1}$$

$$\nabla.\mathbf{u} = 0, \tag{2.2}$$

$$\nabla.\mathbf{B} = 0, \tag{2.3}$$

$$\partial \mathbf{B}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}), \tag{2.4}$$

where $\mathbf{u}$ represents the velocity of the fluid relative to the rotating frame, t time, ρ density, p the pressure in excess of that required to balance the centrifugal force, μ magnetic permeability and $\mathbf{B}$ magnetic field (Shercliff 1965; Hide 1969*a*). The equilibrium state $\mathbf{u} = 0$, $\mathbf{B} = \mathbf{B}_0$ is an exact solution of these equations. We perturb this basic state by small amounts $\mathbf{u}$ and $\mathbf{b}$ respectively, linearize the equations in the usual way and seek solutions in which all perturbation quantities ψ may be written

$$\psi = \mathscr{R}[\hat{\psi}(r)e^{i(m\theta+nz-\omega t)}]. \tag{2.5}$$

We thus find

$$\hat{b}_r = -\frac{\hat{u}_r}{\omega}\left(nB_z + \frac{mB_\theta}{r}\right), \tag{2.6}$$

$$\hat{b}_\theta = \frac{(\hat{u}_r B_\theta)'}{i\omega} - \frac{n}{\omega}(\hat{u}_\theta B_z - \hat{u}_z B_\theta), \tag{2.7}$$

$$\hat{b}_z = \frac{(r\hat{u}_r B_z)'}{ri\omega} + \frac{m(\hat{u}_\theta B_z - \hat{u}_z B_\theta)}{r\omega}, \tag{2.8}$$

$$\hat{u}_z = -\frac{(r\hat{u}_r)'}{rin} - \frac{m\hat{u}_\theta}{rn}, \tag{2.9}$$

$$\left(1 + \frac{m^2}{r^2n^2}\right) ri\hat{u}_\theta = -\frac{m}{rn^2}(r\hat{u}_r)' - \frac{\hat{u}_r}{P}\left\{\frac{2\Omega r}{\omega} + \frac{2nV_\theta}{\omega^2}\left(V_z + \frac{mV_\theta}{rn}\right)\right\}, \tag{2.10}$$

where

$$P(r) = \left(V_z + \frac{mV_\theta}{rn}\right)^2 \frac{n^2}{\omega^2} - 1, \tag{2.11}$$

$$V_\theta(r) = \frac{B_\theta(r)}{(\mu\rho)^{\frac{1}{2}}}, \quad V_z(r) = \frac{B_z(r)}{(\mu\rho)^{\frac{1}{2}}}, \tag{2.12}$$

and primes denote differentiation with respect to r. It is convenient at this stage to replace $\hat{u}_r(r)$ by the more economical symbol $u(r)$, which then satisfies the following differential equation:

$$Pu'' + \left\{P' + \frac{P}{r}\left[\frac{r^2+3m^2/n^2}{r^2+m^2/n^2}\right]\right\}u' + Gu = 0, \tag{2.13}$$

where
$$G(r) = \frac{n^2}{\omega^2}\left\{\frac{2V_z V_z'}{r} + r\left(\frac{V_\theta^2}{r^2}\right)'\right\} + 4\left[\frac{\Omega}{\omega} + \left(V_z + \frac{mV_\theta}{rn}\right)\frac{nV_\theta}{r\omega^2}\right]^2 \frac{n^2}{P}$$
$$+ 4\left[\frac{\Omega}{\omega} + \left(V_z + \frac{mV_\theta}{rn}\right)\frac{nV_\theta}{r\omega^2}\right]\frac{m}{r^2+m^2n^{-2}}$$
$$- \frac{P}{r^2+m^2n^{-2}}\left\{r^2n^2 + 1 + 2m^2 + \frac{m^2}{n^2r^2}(m^2-1)\right\}. \tag{2.14}$$

Subsequent sections will be concerned with the properties of this equation subject to the boundary conditions of no flow through the container walls, i.e.

$$u(r_1) = u(r_2) = 0.$$

In view of the geophysical motivation for the study of this problem we shall be interested primarily in non-axisymmetric disturbances.

3. Azimuthal propagation of non-axisymmetric unstable modes

Equation (2.13) may be written

$$\left\{\frac{r^3Pu'}{r^2+m^2n^{-2}}\right\}' + \frac{r^3Gu}{r^2+m^2n^{-2}} = 0, \tag{3.1}$$

where
$$P(r) = \left(V_z + \frac{mV_\theta}{rn}\right)^2 \Big/ c^2 - 1, \quad (c = \omega/n). \tag{3.2}$$

Multiplying (3.1) by the complex conjugate of u and integrating between the boundaries $r = r_1$ and $r = r_2$ (at which u must vanish) we find

$$\int_{r_1}^{r_2} \frac{r^3}{r^2+m^2n^{-2}}\{G(r)|u|^2 - P(r)|u'|^2\}dr = 0. \tag{3.3}$$

Now let $\omega = \omega_r + i\omega_i$ (subscripts denoting real and imaginary parts) and multiply (3.3) by c^2. Equating the real and imaginary parts of the left-hand side separately to zero we conclude (from the imaginary part) that

$$2c_rc_i\int_{r_1}^{r_2}\frac{r^3}{r^2+m^2n^{-2}}\left\{|u'|^2 + |u|^2\left[S_1(r) + \frac{2\Omega m}{(r^2+m^2n^{-2})nc_r} + \frac{4S_2(r)}{r^2|c^2P|^2}\right]\right\}dr = 0, \tag{3.4}$$

where
$$S_1(r) \equiv \frac{1}{r^2+m^2n^{-2}}\left\{r^2n^2 + 1 + 2m^2 + \frac{m^2}{n^2r^2}(m^2-1)\right\}, \tag{3.5}$$

and
$$S_2(r) \equiv \left\{\Omega rc_r + \left(V_z + \frac{mV_\theta}{rn}\right)V_\theta\right\}^2 - \Omega^2r^2c_i^2$$
$$+ \frac{\Omega r}{c_r}\left\{\Omega rc_r + \left(V_z + \frac{mV_\theta}{rn}\right)V_\theta\right\}\left\{\left(V_z + \frac{mV_\theta}{rn}\right)^2 - c_r^2 + c_i^2\right\}. \tag{3.6}$$

In the absence of rotation ($\Omega = 0$) the integrand in (3.4) is positive throughout the interval $r_1 \leqslant r \leqslant r_2$ (since the azimuthal wavenumber m may take only integral values), the integral therefore cannot vanish, and we conclude that $c_r c_i = 0$ so that any unstable modes do not propagate. In a rotating fluid, on the other hand, non-axisymmetric disturbances may both grow in amplitude and propagate, and we now turn attention to such modes, for which $c_r c_i \neq 0$.

First note that $S_1(r)$ is always positive and that

$$S_2(r) \equiv \left(V_z + \frac{mV_\theta}{rn}\right)\left\{\left(V_z + \frac{mV_\theta}{rn}\right)(V_\theta^2 + \Omega^2 r^2) + \frac{\Omega V_\theta r}{c_r}\left[\left(V_z + \frac{mV_\theta}{rn}\right)^2 + c_r^2 + c_i^2\right]\right\}. \quad (3.7)$$

Unless the inequality

$$\frac{4S_2(r)}{r^2|c^2P|^2} + \frac{2\Omega m}{(r^2 + m^2 n^{-2})\,nc_r} < 0 \quad (3.8)$$

is satisfied somewhere in the interval $r_1 \leqslant r \leqslant r_2$ the integrand in (3.4) is everywhere positive, the integral cannot vanish, and with our initial assumption $c_r c_i \neq 0$ we are led to a contradiction. We thus conclude that modes with $c_r c_i \neq 0$ must be such that

$$\frac{4}{r^2}\left(V_z + \frac{mV_\theta}{rn}\right)\left\{\left(V_z + \frac{mV_\theta}{rn}\right)(V_\theta^2 + \Omega^2 r^2) + \frac{\Omega V_\theta r}{c_r}\left[\left(V_z + \frac{mV_\theta}{rn}\right)^2 + c_r^2 + c_i^2\right]\right\} + \frac{2\Omega m}{nc_r(r^2 + m^2 n^{-2})}\left\{\left[\left(V_z + \frac{mV_\theta}{rn}\right)^2 - c_r^2 + c_i^2\right]^2 + 4c_r^2 c_i^2\right\} < 0 \quad (3.9)$$

somewhere in the interval $r_1 \leqslant r \leqslant r_2$.

If the magnetic field is purely azimuthal then (3.9) becomes

$$\frac{4}{r^2}\left\{\frac{m^2 V_\theta^2}{r^2 n^2}(V_\theta^2 + \Omega^2 r^2) + \frac{V_\theta^2 \Omega m}{nc_r}\left(\frac{m^2 V_\theta^2}{r^2 n^2} + c_r^2 + c_i^2\right)\right\} + \frac{2\Omega m}{nc_r(r^2 + m^2 n^{-2})}\left\{\left(\frac{m^2 V_\theta^2}{r^2 n^2} - c_r^2 + c_i^2\right)^2 + 4c_r^2 c_i^2\right\} < 0, \quad (3.10)$$

whence it is clear that, *regardless of the details of the magnetic field profile*, any unstable disturbances must have

$$\frac{\Omega m}{nc_r} = \frac{\Omega m}{\omega_r} \equiv \frac{\Omega}{c_{\theta r}} < 0, \quad (3.11)$$

where $c_{\theta r} = \omega_r/m$ is the phase velocity in the azimuthal direction, *and must therefore propagate against the basic rotation, i.e. 'westward'*. Comparing (3.9) and (3.10) it is also clear that this result still holds even if the magnetic field varies in both magnitude *and* direction with distance from the rotation axis provided that

$$|V_z| \leqslant \left|\frac{mV_\theta}{rn}\right| \quad (3.12)$$

everywhere in the interval $r_1 \leqslant r \leqslant r_2$. If the axial and azimuthal dimensions of an unstable disturbance are comparable it must therefore propagate westward, provided only that the axial magnetic field is somewhat less than the azimuthal field everywhere. Finally note that all unstable disturbances do in fact propagate westward when the magnetic field is purely *axial*, as evinced by setting $V_\theta = 0$ in (3.9).

Thus hydromagnetic waves generated in a rotating fluid by instabilities due to spatial variations in the magnetic field propagate westward for a wide range of magnetic field profiles. These profiles must, however, possess a few simple properties for there to be any unstable modes at all, and it is to these properties that we now turn attention.

4. Sufficient conditions for stability: azimuthal magnetic field

We first note certain conditions under which unstable non-axisymmetric modes *must* propagate (in contrast to the non-rotating case). Supposing that they do not (i.e. $c_r = 0$), (3.4) becomes

$$\Omega c_i \int_{r_1}^{r_2} \frac{r^3|u|^2}{r^2+m^2n^{-2}} \left\{ \frac{2(V_z+mV_\theta/rn)\,V_\theta}{r[(V_z+mV_\theta/rn)^2+c_i^2]} + \frac{m}{n(r^2+m^2n^{-2})} \right\} dr = 0. \tag{4.1}$$

We thus obtain a contradiction (for the left-hand side cannot then vanish) if either (*a*) the field is purely azimuthal, (*b*) the field is purely axial or (*c*) both components are present but $|V_z| < |mV_\theta/(rn)|$ everywhere in the interval $r_1 \leqslant r \leqslant r_2$, and accordingly learn that under any of these three conditions non-axisymmetric unstable disturbances in a rotating fluid must propagate. We confine attention in the remainder of this section to the case in which the field is entirely azimuthal. When investigating non-axisymmetric unstable modes it is therefore appropriate to take both c_r and c_i as non-zero. In this case the imaginary part of (3.3) becomes

$$\frac{2\omega_r\omega_i}{|\omega|^2} \int_{r_1}^{r_2} \frac{r^3}{r^2+m^2n^{-2}} \left[\frac{V_\theta^2 m^2}{r^2|\omega|^2} |u'|^2 + S_3(r)|u|^2 \right] dr = 0, \tag{4.2}$$

where, after some manipulation, $S_3(r)$ may be written as

$$\begin{aligned} S_3(r) = {} & \frac{4n^2\Omega^2|\omega|^2}{|\omega^2 P|^2} - \frac{n^2 r}{|\omega|^2}\left(\frac{V_\theta^2}{r^2}\right)' \\ & - \frac{2\Omega m}{\omega_r}\left\{ \frac{2n^2 V_\theta^2}{|\omega^2 P|^2 r^2}\left[\frac{V_\theta^2 m^2}{r^2} - 3\omega_r^2 + \omega_i^2\right] + \frac{1}{r^2+m^2n^{-2}} \right\} \\ & + \frac{m^2 V_\theta^2}{r^2|\omega|^2}\left\{ S_1(r) - \frac{4}{r^2+m^2n^{-2}} - \frac{4n^2}{m^2}\left[1 + \frac{r^2|\omega|^2}{V_\theta^2 m^2}\left(\frac{V_\theta^2 m^2}{r^2} - 2\omega_r^2 + 2\omega_i^2\right)^{-1}\right]^{-1} \right\}, \end{aligned} \tag{4.3}$$

and
$$P(r) = \frac{V_\theta^2 m^2}{r^2\omega^2} - 1.$$

Clearly if $m = 0$ then $P = -1$ and

$$S_3(r) = \frac{n^2}{|\omega|^2}\left\{ 4\Omega^2 - r\left(\frac{V_\theta^2}{r^2}\right)' \right\}. \tag{4.4}$$

Inspection of (4.2) then reveals that if

$$L \equiv \frac{4\Omega^2}{r} - \left(\frac{V_\theta^2}{r^2}\right)' \tag{4.5}$$

does not change sign in the interval then either ω_r or ω_i must be zero. Michael (1954) was in fact able to show that the system is stable to axisymmetric disturbances *if and only if* $L > 0$ everywhere in the interval $r_1 \leqslant r \leqslant r_2$. Thus even if the magnetic field configuration is such as to promote instability the system can always be rendered stable to axisymmetric disturbances by sufficiently rapid rotation.

As far as non-axisymmetric disturbances are concerned we know from the previous section that all unstable modes drift west; so consider here $\Omega m\omega_r < 0$. If $V_\theta^2 m^2 > 3r^2\omega_r^2$ everywhere in the interval the second term on the right-hand side of (4.3) will be positive. The final term exceeds

$$\frac{m^2V_\theta^2}{r^2|\omega|^2}\left\{S_1(r)-\frac{4}{r^2+m^2n^{-2}}-\frac{4n^2}{m^2}\right\}$$
$$\equiv \frac{m^2V_\theta^2}{r^2|\omega|^2(r^2+m^2n^{-2})}\left\{n^2r^2\left(1-\frac{4}{m^2}\right)+(2m^2-7)+\frac{m^2}{r^2n^2}(m^2-1)\right\}, \tag{4.6}$$

which for $|m| > 1$ (and, of course, integral) is patently positive. Inspection of (4.2) then leads to the conclusion that unstable non-axisymmetric modes, such that $V_\theta^2 m^2 > 3r^2\omega_r^2$ everywhere and $|m| > 1$, can only occur *if V_θ^2/r^2 increases with radius somewhere in the interval* $r_1 \leqslant r \leqslant r_2$.

5. Local stability analysis: azimuthal magnetic field

It is difficult to gain further insight into the instability mechanism while maintaining the generality of the previous two sections. We accordingly now present a *local* analysis (following Velikhov 1959; Schubert 1968) in which, while still not restricting attention to any particular magnetic field profile, we investigate the stability of disturbances with radial wavelengths which are short compared with the natural length scale associated with spatial variations in the magnetic field. Such an analysis brings out very clearly essential differences between axisymmetric and non-axisymmetric disturbances when the fluid is in rapid rotation (cf. the non-hydromagnetic problem discussed, for example, by Joseph & Munson (1970)).

Axisymmetric disturbances

Equation (2.13) may then be written as

$$u''+\frac{u'}{r}-\left\{n^2+\frac{1}{r^2}+\frac{n^2}{\omega^2}\left[r\left(\frac{V_\theta^2}{r^2}\right)'-4\Omega^2\right]\right\}u = 0. \tag{5.1}$$

Now consider the local solutions of (5.1) in the neighbourhood of a particular radius $r = r_0$ so that the coefficients may be regarded as uniform (to a first approximation) in that neighbourhood. The equation then admits solutions $u \propto e^{ilr}$, where l is a local radial wavenumber satisfying

$$l^2-\frac{il}{r_0}+n^2+\frac{1}{r_0^2}+\frac{n^2}{\omega^2}\left[r_0\left(\frac{V_\theta^2}{r^2}\right)'_{r=r_0}-4\Omega^2\right] = 0. \tag{5.2}$$

This local analysis will only be justified, however, if the radial wavelength is small compared with the radius, i.e. if $l \gg 2\pi/r_0$, and we must therefore for consistency neglect the second and fourth terms in (5.2) compared with the first, whence

$$\omega^2 = \frac{\{4\Omega^2 - r_0(V_\theta^2/r^2)'_{r=r_0}\}\, n^2}{l^2+n^2}. \tag{5.3}$$

The system is therefore stable to axisymmetric disturbances if and only if

$$\frac{4\Omega^2}{r_0} > \left(\frac{V_\theta^2}{r^2}\right)'_{r=r_0}, \tag{5.4}$$

and this is, of course, consistent with the studies of Michael (1954).

Non-axisymmetric disturbances

Setting $V_z = 0$ in (2.13) and applying the same procedure to that equation (assuming $P'(r_0) \sim P(r_0)/r_0$ and $l \gg 2\pi/r_0$) we find

$$\frac{4n^2}{P(r_0)}\left[\frac{\Omega}{\omega}+\frac{m}{\omega^2}\left(\frac{V_\theta^2}{r^2}\right)_{r=r_0}\right]^2 + \frac{4m}{r_0^2+m^2n^{-2}}\left[\frac{\Omega}{\omega}+\frac{m}{\omega^2}\left(\frac{V_\theta^2}{r^2}\right)_{r=r_0}\right] - \left(l^2+n^2+\frac{m^2}{r_0^2}\right)P(r_0) + \frac{r_0 n^2}{\omega^2}\left(\frac{V_\theta^2}{r^2}\right)'_{r=r_0} = 0. \tag{5.5}$$

Our main interest lies in the 'slow' hydromagnetic waves that propagate in a 'rapidly' rotating fluid (see §1). We do not yet know the growth rate of such waves, however, so that while it is helpful at this stage to replace

$$P = m^2V_\theta^2/r^2\omega^2 - 1 \quad \text{by} \quad m^2V_\theta^2/r^2\omega^2$$

we shall have to justify this step *a posteriori*. Equation (5.5) then becomes a quadratic equation for ω with roots given by

$$\frac{2\omega\Omega}{m(V_\theta^2/r^2)_{r=r_0}} = -2 - \frac{m^2}{n^2(r_0^2+m^2n^{-2})} \pm\left\{m^2\left[1+\frac{m^2}{r_0^2n^2}+\frac{l^2}{n^2}+\frac{m^2}{n^4(r_0^2+m^2n^{-2})^2}\right] - \frac{r_0(V_\theta^2/r^2)'_{r=r_0}}{(V_\theta^2/r^2)_{r=r_0}}\right\}^{\frac{1}{2}}. \tag{5.6}$$

Noting that

$$m^2\left[1+\frac{m^2}{r_0^2n^2}+\frac{l^2}{n^2}+\frac{m^2}{n^4(r_0^2+m^2n^{-2})^2}\right] > 4+\frac{4m^2}{n^2(r_0^2+m^2n^{-2})}+\frac{m^4}{n^4(r_0^2+m^2n^{-2})^2}, \tag{5.7}$$

if $|m| > 1$, we conclude that when $(V_\theta^2/r^2)'_{r=r_0} \leqslant 0$ all modes with $|m| > 1$ are stable and may propagate both east and west. If $(V_\theta^2/r^2)'$ is positive and sufficiently large, however, then unstable modes will result and such waves propagate *westward* in accord with the very much more general conclusions of §3.

For unstable modes we clearly require

$$\frac{r_0(V_\theta^2/r^2)'_{r=r_0}}{(V_\theta^2/r^2)_{r=r_0}} > \frac{m^2l^2}{n^2}, \tag{5.8}$$

and since (for a local analysis to be appropriate) we must have $l^2 \gg r_0^{-2}$, (5.8) can only be satisfied if $m^2/n^2r_0^2 \ll 1$ (assuming $r_0(V_\theta^2/r^2)'_{r=r_0} \sim (V_\theta^2/r^2)_{r=r_0}$). The character of the unstable waves is therefore displayed by a somewhat simplified version of (5.6):

$$\frac{2\omega\Omega}{m(V_\theta^2/r^2)_{r=r_0}} = -2 \pm\left\{m^2\left(1+\frac{l^2}{n^2}\right) - \frac{r_0(V_\theta^2/r^2)'_{r=r_0}}{(V_\theta^2/r^2)_{r=r_0}}\right\}^{\frac{1}{2}}. \tag{5.9}$$

This formula shows that the growth rate of the unstable waves will in general be comparable with their frequency and that $|\omega| \sim |mV_\theta^2/\Omega r^2|$. Our initial assumption that P may be replaced to good approximation by $m^2V_\theta^2/r^2\omega^2$ is therefore valid provided $\Omega^2 \gg (V_\theta^2/r^2)_{r=r_0}$.

Equation (5.9) shows how the destabilizing effect of a magnetic field *gradient* such that $(V_\theta^2/r^2)' > 0$ is opposed by the stabilizing effect of the field itself, for the more the lines of force are twisted (i.e. the higher the value of m) the greater the restoring force of the 'equivalent elastic strings'. It is also natural that disturbances with large radial wavenumbers l are less likely to be unstable than those with small ones, for in view of their smaller dimensions they will 'feel' the destabilizing influence of a radial magnetic field gradient correspondingly less.

While this local analysis has therefore provided some insight into the nature of the unstable modes we must bear in mind how the conclusion that they must be characterized by very low values of $m^2/n^2r_0^2$, i.e. by axial rather than azimuthal propagation, is a direct consequence of the assumptions required for the validity of such an analysis. This 'weak' azimuthal propagation will not necessarily be characteristic of the more general circumstances covered by the analyses of the earlier sections.

6. Discussion

Sections 4 and 5 have made it clear that while a 'rapidly' rotating fluid annulus permeated by an azimuthal magnetic field is stable to axisymmetric disturbances unless

$$\frac{r(V_\theta^2/r^2)'}{V_\theta^2/r^2} > \frac{4\Omega^2r^2}{V_\theta^2} \gg 1 \tag{6.1}$$

somewhere (cf. equation (4.5)), the amplitude of non-axisymmetric disturbances can grow with time even if

$$\frac{r(V_\theta^2/r^2)'}{(V_\theta^2/r^2)} \gtrsim O(m^2) \tag{6.2}$$

(cf. equation (5.9)). Thus when $\Omega^2r^2 \gg V_\theta^2$ non-axisymmetric modes are more likely to be unstable than axisymmetric ones. The reason is that axisymmetric disturbances do not twist the lines of force of an azimuthal magnetic field and as the rotation speed increases indefinitely the Lorentz force becomes negligible in comparison with the Coriolis force so that hydromagnetic effects are masked (as $\Omega \to \infty$ equation (5.3), for example, approaches the dispersion relationship for axisymmetric inertial waves). Non-axisymmetric disturbances, however, twist the lines of force and accordingly take the form, at 'rapid' rotation speeds, of 'slow' hydromagnetic waves (see § 1). It is a fundamental property of such waves that they are characterized by a delicate balance between the vorticity induced by Lorentz and Coriolis forces, and this persistent balance discourages the tendency toward two-dimensionality (and therefore stability) that would prevail if hydromagnetic effects were absent. Twisting of the field lines therefore produces an intimate coupling between hydromagnetic and rotational effects that prevents the former from being masked however small the magnetic field.

This persistent coupling is due to the perfect conductivity of the fluid, for

Alfvén's classic theorem concerning the 'freezing-in' of lines of force to the fluid follows directly from that assumption, and *the strength of the attachment of the lines of force to the fluid is then in no way dependent on the strength of the magnetic field.* While effects due to finite conductivity have not yet been fully investigated, elementary considerations (Acheson 1971) suggest that for the above analysis to be appropriate the 'hydromagnetic interaction parameter'

$$\alpha \equiv V^2\sigma\mu/2\Omega$$

must be much greater than unity, where σ is the electrical conductivity of the fluid. This parameter (like Q) is one of the most important to emerge so far from studies of the magnetohydrodynamics of rotating fluids. In addition to acting (inversely) as a measure of the decay of 'slow' hydromagnetic waves due to ohmic dissipation (Acheson 1971) it indicates the relative importance of hydromagnetic and rotational effects on viscous boundary layers (e.g. Gilman & Benton 1968; Benton & Loper 1969; Hide 1969*b*; Loper & Benton 1970).

To summarize, then, it appears that a non-uniform magnetic field such that $(B_\theta^2/r^2)' > 0$ somewhere can act as a westward selection mechanism for 'slow' hydromagnetic waves, for only westward-propagating waves can feed on some 'available magnetic energy' in the basic magnetic field distribution and grow in amplitude with time. This mechanism is evidently insensitive to the details of the magnetic field profile and even to changes in *direction* as well as magnitude of the imposed field (see equation (3.12)). Although it is not evident that spherical, rather than cylindrical boundaries will change the main conclusions of this paper, a corresponding analysis in a thick spherical shell is naturally desirable, particularly in view of the possible role of these hydromagnetic instabilities in the origin of the westward drift of the geomagnetic field. In conclusion, however, we remark that no azimuthal magnetic field varying with latitude in a *thin* spherical shell gives rise to these instabilities (Acheson (1971), see also Gilman (1967) for a 'β plane' analysis including baroclinic effects). How this bears on the present problem is far from clear, for the assumptions in that analysis of (i) essentially two-dimensional disturbances, and (ii) no radial (in the spherical sense) velocity or magnetic field components, rendered the system highly constrained in comparison with that studied above. It will, however, be discussed more fully in a future paper along with some related results providing further clues to the precise physical nature of these hydromagnetic instabilities. It appears from a preliminary analysis than an investigation of 'slow' wave generation by an unstable *density* gradient (certain aspects of which have been discussed by Braginsky (1967)) may help a great deal in this respect.

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Appendix

We finally include some consideration of the energetics of the above system when it is not open-ended but bounded at $z = z_1$ and $z = z_2$. (To satisfy the additional boundary conditions $u_z(r, \theta, z_1) = u_z(r, \theta, z_2) = 0$ the axial wavenumber n will then take a discrete set of values such that $n(z_2 - z_1)$ is an integral multiple of π.)

From equations (2.1)–(2.4) it is straightforward (see, for example, Roberts 1967) to derive the energy equation:

$$\frac{\partial}{\partial t}\iiint\left(\tfrac{1}{2}\rho\mathbf{u}^2+\frac{1}{2\mu}\mathbf{B}^2\right)d\tau = -\frac{1}{\mu}\iint(\mathbf{E}\wedge\mathbf{B}).d\mathbf{S} \tag{A 1}$$

$$=\frac{1}{\mu}\iint(\mathbf{B}.\mathbf{u})\,\mathbf{B}.d\mathbf{S}, \tag{A 2}$$

where $\mathbf{E}$ (which, to prevent the flow of infinite currents in a perfectly conducting fluid, must be equal to $-\mathbf{u}\wedge\mathbf{B}$) is the electric field, $d\tau$ denotes an element of volume and $d\mathbf{S}$ denotes a directed element of area of the bounding surface. This equation amounts to the statement that the total energy (i.e. kinetic + magnetic) of the system discussed in this paper can increase only as a result of a Poynting flux of electromagnetic energy through the boundaries. We now show that to a first approximation this flux is zero and that the total energy therefore remains constant.

Write $\mathbf{B}$ as

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}_l + \mathbf{b}_R, \tag{A 3}$$

where $\mathbf{B}_0 = \{0, B_\theta(r), 0\}$ is the imposed azimuthal magnetic field, $\mathbf{b}_l$ is that part of the magnetic field perturbation satisfying *exactly* the linearized equations of motion (2.6)–(2.14) and $\mathbf{b}_R$ is the resultant of all the higher order terms in a formal expansion of $\mathbf{B}$ in powers of some typical value ϵ of $|\mathbf{b}_l|/|\mathbf{B}_0|$ (thus $|\mathbf{b}_R|/|\mathbf{b}_l|$ is of order ϵ). Equations (2.6) and (2.8) then show that since the normal component of $\mathbf{u}$ vanishes at both the cylindrical and top and bottom boundaries so does the normal component of $\mathbf{b}_l$. On inserting the expansion (A 3) into (A 2) this observation leads us to conclude, to a first approximation, that

$$\frac{\partial}{\partial t}\iiint\left(\tfrac{1}{2}\rho\mathbf{u}^2+\frac{1}{2\mu}\mathbf{B}^2\right)d\tau = 0, \tag{A 4}$$

so that there is no interchange of energy between the fluid and its surroundings and the sole energy source for the wave generation thus lies in the basic magnetic field.

We now inquire how the total energy of the *wave* changes with time. Denoting the perturbation field $\mathbf{b}_l + \mathbf{b}_R$ by $\mathbf{b}$, we find (on using (2.4), some vector identities, and the fact that the normal component of $\mathbf{b}_l$ vanishes at the boundaries) that

$$\frac{\partial}{\partial t}\iiint\left(\tfrac{1}{2}\rho\mathbf{u}^2+\frac{1}{2\mu}\mathbf{b}^2\right)d\tau = -\frac{1}{\mu}\iiint(\nabla\wedge\mathbf{B}_0).(\mathbf{u}\wedge\mathbf{B})\,d\tau \tag{A 5}$$

$$= -\frac{1}{\mu}\iiint(\nabla\wedge\mathbf{B}_0).(\mathbf{u}_l\wedge\mathbf{b}_l+\mathbf{u}_R\wedge\mathbf{B}_0)\,d\tau. \tag{A 6}$$

(In view of the fact that

$$\mathbf{u}_l = \mathbf{u}_1(r)\cos(m\theta+nz-\omega t)+\mathbf{u}_2(r)\sin(m\theta+nz-\omega t)$$

and m is an integer, the leading term

$$\mu^{-1}\iiint(\nabla\wedge\mathbf{B}_0).(\mathbf{u}_l\wedge\mathbf{B}_0)\,d\tau$$

is zero.) Equation (A 5) indicates the importance of a magnetic field *gradient* as far as wave generation is concerned.

Not only is it clear that no energy is available for such generation in the no-current case $\nabla\wedge\mathbf{B}_0 = 0$, i.e. $B_\theta \propto r^{-1}$ (in keeping with the necessary condition derived above, namely $(B_\theta^2/r^2)' > 0$ somewhere), but it becomes clear energetically why no *slow* wave generation is possible due to a *constant* current (which is, incidentally, in keeping with the results of Malkus (1967*a*) for a *sphere*). To see this, note that the slow waves are characterized by very small values of $|\omega|/|\mathbf{\Omega}|$ (see equation (5.6)) so that (2.1) may be written as

$$2\mathbf{\Omega}\wedge\mathbf{u} = -\frac{1}{\rho}\nabla p+\frac{1}{\mu\rho}(\nabla\wedge\mathbf{B})\wedge\mathbf{B} \tag{A 7}$$

with error $O(V^2/\Omega^2R^2)$ (if the wavelengths involved are comparable with the outer radius of the annulus R). Taking the scalar product of (A 7) with $\mathbf{B}$ and integrating over the volume of the cylinder we thus find

$$\iiint 2\mathbf{\Omega}.(\mathbf{u}\wedge\mathbf{B})d\tau = -\frac{1}{\rho}\iint p\mathbf{b}_R.d\mathbf{S}. \tag{A 8}$$

Observe from (A 7) that $p/\rho R|\mathbf{\Omega}||\mathbf{u}_l|$ is at most of order unity, so that if the (axial) current $\mathbf{j}_0 = \mu^{-1}\nabla\wedge\mathbf{B}_0$ is constant we find from (A 8) that

$$\mu^{-1}\iiint(\nabla\wedge\mathbf{B}_0).(\mathbf{u}\wedge\mathbf{B})\,d\tau$$

is at most of order $(\mu^{-1}u_lBR^2)b_R$. But note from (A 5) and (A 6) that, in any other event, it is typically of order $(\mu^{-1}u_lBR^2)b_l$! Thus we see, on energetic grounds, the importance of a radial gradient of electric current.

We finally remark that, while hydromagnetic waves in a non-rotating fluid are characterized by equipartition between kinetic and magnetic energy, such equipartition does not persist when the fluid is rotating. Indeed we may easily see by setting $\omega \sim V^2/\Omega R^2$ in (2.6), (2.7) or (2.8) that the kinetic energy of 'slow' waves in a rapidly rotating fluid ($\Omega R \gg V$) accounts for only a very small fraction ($\sim V^2/\Omega^2R^2$) of the total wave energy, the remainder being in the form of magnetic energy.

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